Fast Simulation of Non-Linear Mechanical Systems

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NOMENCLATURE

B	:	System Impedance Matrix
C	:	Damping Matrix
H	:	System Transfer Function Matrix
K	:	Stiffness Matrix
M	:	Mass Matrix
N	:	Number of Modes
R	:	Residue
f	:	Input Force Vector
g	:	Single Non-Linear Component
n	:	Discrete Time Index
r	:	Mode Number
s	:	Laplace Domain Variable
z	:	z Domain Variable
ζ	:	Damping Ratio

ABSTRACT

In the process of Shock Response Spectrum calculation, for instance, the forced response of a mechanical Single Degree of Freedom (SDOF) system is determined. This is in most cases performed with the use of a digital filter, preferably one designed with the ramp invariance method. If the sampling frequency is selected properly, a very high precision may be reached. The error may also be predicted. The method may be extended to forced response calculation for mechanical systems of virtually any degree of freedom by restricting the calculations to the frequency range of interest. The parameters of the mechanical system may come from analytic expressions, FEM modeling or from experimental data. The method also allows the incorporation on non-linear elements in the system. This leads to simulations of forced response which are much faster and more accurate than those performed in most commercial software.

1. INTRODUCTION

The need for fast simulations especially in real-time applications is a most critical factor, thus the ever increasing interest from the scientific and engineering community in developing fast and robust algorithms. In the vibration industry, modeling of non-linearity in structures is of growing concern as we seek more and more to expand the structural performance of mechanical systems. Existing techniques such as the reverse path method [1] can lead to ill-conditioned PSD matrices, which poses accuracy problems.

A previous paper co-authored by Kjell Ahlin [2] looked into the use of the ramp invariance digital method in the transient forced response of linear mechanical structures. One common application is the shock response spectrum (SRS) calculation, where the forced response of a mechanical SDOF system is to be determined. A typical model for the SRS calculation, shown in Figure 1, assumes that the signal to be analyzed is applied to an array of independent SDOF systems at a common base and of interest then is the peak response of each of the masses.

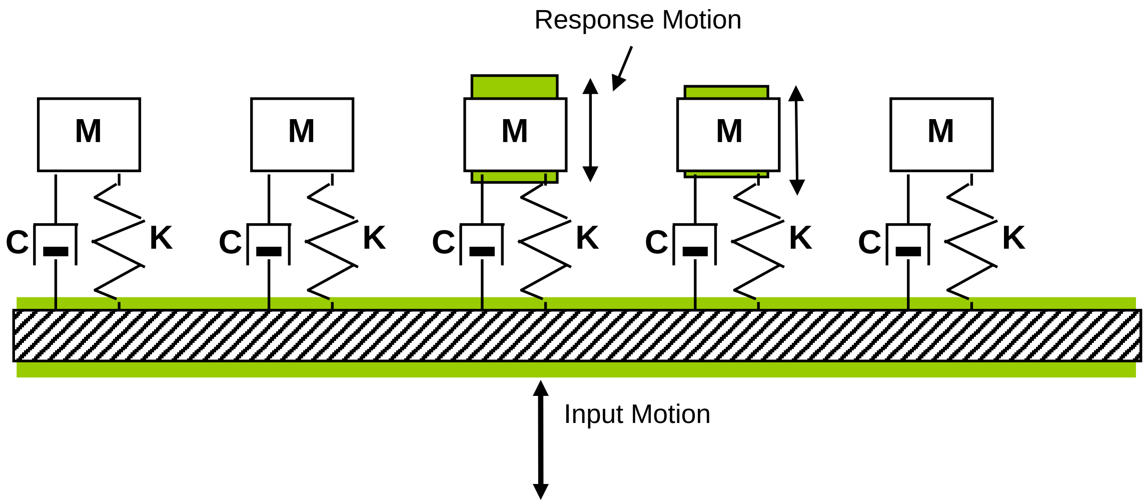

Figure 1. Typical model used for the calculation of the shock response spectrum. The motion is assumed to be applied to a common base of an array of damped and independent SDOF systems.

This paper has as principal objective extending the afore-mentioned work by incorporating a single non-linear component in the system. The paper is organized as follows: After a brief review of common time domain integration methods in Section 2, a digital filter method based on the ramp invariance method is considered in Section 3. Here, three different Multiple Degree of Freedom (MDOF) systems will be considered involving different configurations regarding the location of the non-linear component. And in section 4 are presented some computer simulations.

2. TIME DOMAIN INTEGRATION METHODS

There exist several methods to obtain the response to arbitrary excitation. A few of the time domain methods are: the Duhamel integral, the state transition matrix method, and the Runge-Kutta method and with variations. The Duhamel integral, also known as the convolution integral, is a well-established method in the literature, e.g. see [3]; its main drawback is the computational burden associated with its nonrecursive nature. A second method, the state transition matrix method, is based on re-arranging the second order equations of motion into first order ones and applying the Duhamel integral technique to solve for the response. However, the integral in this case can be used to derive a recursive algorithm which leads to a faster solution than the naive Duhamel integral method [3]. It is to be noted that the state transition matrix method is also known as the step invariance method. And one last method of interest is the popular Runge-Kutta family methods, which are very easy to implement on a computer and generally offer good accuracy with a small step size.

However the task of choosing a proper step size in the integration methods is a delicate one and repeatedly solving the problem with half the previous step size is a simple way of ensuring that the desired accuracy level is met. In this context, it should be noted that a variation of the Runge-Kutta method known as the Runge-Kutta-Fehlberg method uses an optimal step size for a desired accuracy. Nonetheless the issue of large computation time still remains for long integration times with the Runge-Kutta family.

3. DIGITAL FILTER METHODS

Working on discrete sampled data necessitates transformation of a model in the continuous time domain into the discrete time domain. The impulse invariance transform, the step invariance transform and the ramp invariance transform are of particular interest in that firstly their frequency dependent error can be computed, secondly they are stable and thirdly they are faster than traditional time domain techniques. For all these digital methods, it is obvious that the frequency dependent error diminishes with increasing sampling frequency.

The impulse invariance transform method is the most popular technique in the literature. It has the advantage of bringing in no phase distortion, but the aliasing phenomenon introduces error in the amplitude function of the frequency response function, even at zero frequency. The step invariance method (i.e. the state transition matrix method) is more accurate in its amplitude function than the impulse invariance method but there is a phase distortion corresponding to a half time step in its phase characteristic function.

The technique adopted in this paper is the ramp invariance method which has the advantage of exhibiting no phase distortion. However it suffers from an amplitude error which is the square of that of the impulse invariance technique. It is worth noting that the state transition matrix method compares favorably well in terms of computation times with the ramp invariance method.

There are various ways to represent a mechanical system: through a Finite Element Model (FEA) model, a lumped MCK model or an analytical model. All of them can provide quantitative information about the modal parameters i.e. poles and residues of the system. Here we note that these parameters can as well be obtained by performing an experimental modal analysis on the mechanical system.

According to the modal superposition theorem for a MDOF system, consisting of N modes, the transfer system function matrix $H(s)$ can be expressed using partial fraction expansion in terms of the residues R_r and poles s_r as follows:

$$H(s) = \sum_{r=1}^{N} \frac{R_r}{s - s_r} + \frac{R_r^*}{s - s_r^*} \tag{1}$$

On examining Equation (1), it can be seen that the contribution of each mode r in terms of residues and poles corresponds to the transfer function of a second order digital filter, which, in the z-plane, is given by

$$H(z) = \frac{b_0^r + b_1^r \cdot z^{-1} + b_2^r \cdot z^{-2}}{1 + a_1^r \cdot z^{-1} + a_2^r \cdot z^{-2}} \tag{2}$$

where a and b are the filter coefficients.

The interested reader is referred to [4] regarding the procedure of making a ramp invariant filter. The next section shall illustrate how the relationships given in Equations (1) and (2) can be applied to three typical MDOF mechanical systems comprising one non-linear component.

(a). Applied Force and Non-linearity at Same DOF

In the first case, we consider a non-linear MDOF system for which both the applied force and the non-linearity are at the same degree of freedom as shown in Figure 2. More specifically, the single non-linear component is located between the input force DOF and ground.

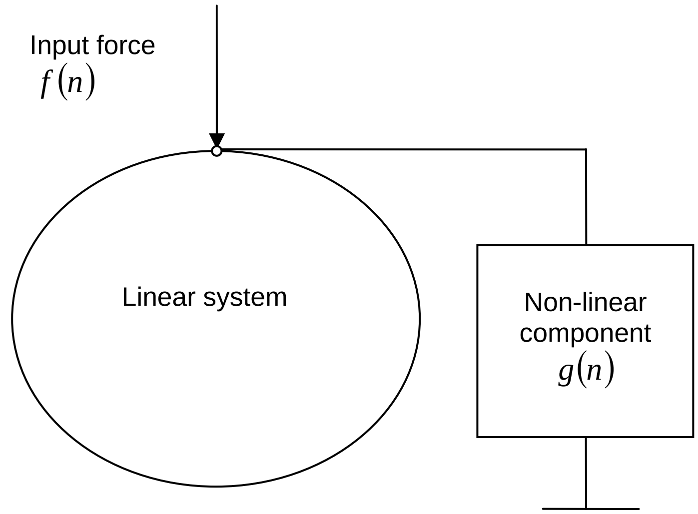

Figure 2. First case configuration: Input force and non-linear component at same DOF.

The equation of motion for the system of interest in discrete time domain is governed by the following equation:

$$M\,\ddot{x}(n)+C\,\dot{x}(n)+K\,x(n)+g(n)=f(n) \tag{3}$$

with the usual notations and where $g(n)$ represents a single non-linear component which is a function of x.

Equation (3) in the frequency domain (z-plane) is conveniently written as

$$B(z)\cdot X(z)+G(z)=F(z) \tag{4}$$

where $B(z)$ is the system impedance matrix and from which equation the system response $X(z)$ can be obtained as

$$X(z)=H(z)\cdot[F(z)-G(z)] \tag{5}$$

Combining Equations (2) and (5) and re-writing the resulting relationship in the time domain gives the system response at time n for the rth mode as

$$\begin{aligned} x_r(n) &= b_0^r\, f(n)-b_0^r\, g(n) \\ &\quad + b_1^r\, f(n-1)-b_1^r\, g(n-1)+b_2^r\, f(n-2)-b_2^r\, g(n-2)-a_1^r\, x_r(n-1)-a_2^r\, x_r(n-2) \end{aligned} \tag{6}$$

Equation (6) is identified as a recursive equation which can be written down for each mode r (i.e. for each digital filter). Using the superposition theorem then, the total response is the sum of each of the individual digital filters.

Taking a step further, Equation (6) can be re-written in a compact fashion for each mode r and at time n as follows:

$$x_r(n)+b_0^r\, g(n)=Q_r \tag{7}$$

where it can be easily seen that the term Q_r embodies the terms in the second line of Equation (6) and also the term $b_0^r\, f(n)$. In this respect we note that Q_r is completely determined by past parameter values together with the current applied force $f(n)$.

Now, Equation (7) can be recognized as being a non-linear equation in x, with g being some non-linear function of x. Equation (7) is computed for each mode r and then summed up for all the modes; after which the resulting equation is solved to yield the desired total response at time step n. To illustrate the procedure just described, consider a MDOF system consisting of 2 modes for which the input-output relationship can be represented schematically as shown in Figure 3.

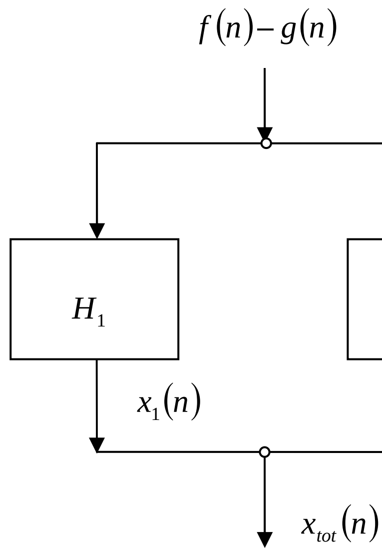

Figure 3. Input-output relationship for a 2-DOF system expressed in terms of an array of digital filters.

To obtain the total response of the system shown in Figure 3, the individual responses x_r $(r = 1, 2)$ are computed from Equation (6) and summed up as follows

$$x_{tot}(n) = x_1(n) + x_2(n) \tag{8}$$

and consideration of Equation (7) together with Equation (8) lead us to the following equation which is required to be solved at each time step:

$$x_{tot}(n) + \left(b_0^1 + b_0^2\right) \cdot g(n) = Q_1 + Q_2 \tag{9}$$

where $g(n)$ is a non-linear function of $x_{tot}(n)$.

(b). Applied Force and Non-linearity at Different DOFs

For the second case, we have an input force at DOF a and a single non-linear component between DOF b and ground as shown in Figure 4.

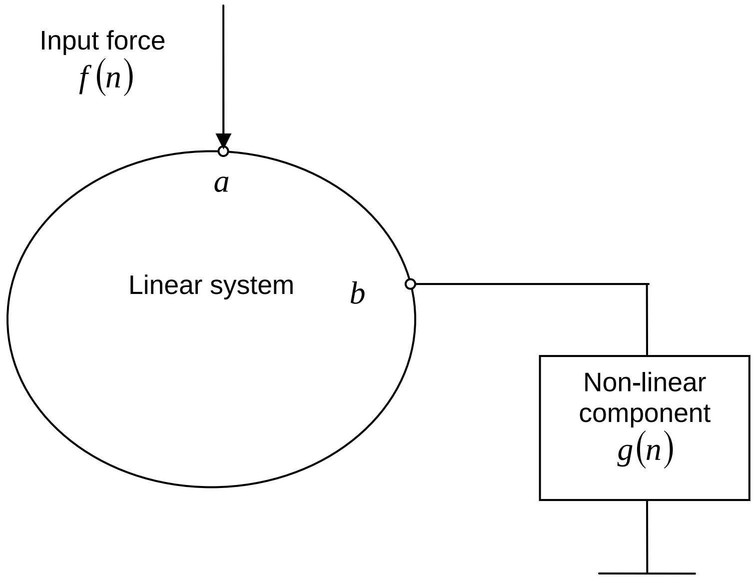

Figure 4. Second case configuration: Input force and non-linear component at different DOFs.

Consideration of Equation (5) enables us to write the following equation for the system response $X(z)$ as

$$\begin{Bmatrix} X_a(z) \\ X_b(z) \end{Bmatrix} = \begin{bmatrix} H_{aa}(z) & H_{ab}(z) \\ H_{ba}(z) & H_{bb}(z) \end{bmatrix} \cdot \begin{Bmatrix} F(z) \\ -G(z) \end{Bmatrix} \tag{10}$$

Of interest in this particular configuration is the response at point b which can be easily calculated from Equation (10). Again, a modal superposition should be applied to obtain the total response.

(c). Applied Force and Non-linearity at Different DOFs, with the Non-Linearity between two other DOFs

In this last case, a schematic of the configuration is shown in Figure 5 where the non-linear component lies between two separate DOFs, excluding the DOF at which the force is applied. Moreover, the non-linear component is assumed to be in the form of

$$g(n) = \phi(x_b - x_c) \tag{11}$$

where ϕ represents a non-linear function in x.

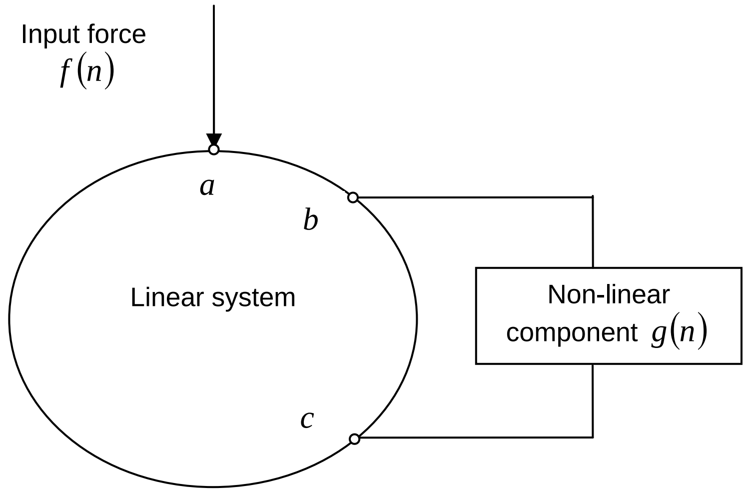

Figure 5. Third case configuration: Input force and non-linear component at different DOFs, with the non-linearity acting between two DOFs, excluding the DOF at which the force is applied.

As shown in part (b) using Equation (5) an expression for the system response at any DOF can be obtained for the present case as

$$\begin{Bmatrix} X_a \\ X_b \\ X_c \end{Bmatrix} = \begin{bmatrix} H_{aa} & H_{ab} & H_{ac} \\ H_{ba} & H_{bb} & H_{bc} \\ H_{ca} & H_{cb} & H_{cc} \end{bmatrix} \begin{Bmatrix} F \\ G \\ -G \end{Bmatrix} \tag{12}$$

Now, since the non-linear component $g(n)$ is located between DOFs b and c, of interest then are the following relationships from Equation (12):

$$X_b = H_{ba}\,F + H_{bb}\,G - H_{bc}\,G \tag{13}$$

$$X_c = H_{ca}\,F + H_{cb}\,G - H_{cc}\,G \tag{14}$$

and

$$(X_b - X_c) = (H_{ba} - H_{ca}) \cdot F + (H_{bb} - H_{cb} - H_{bc} + H_{cc}) \cdot G \tag{15}$$

Thus, with the help of the relationship in Equation (15), the contribution due to the non-linear component can be easily obtained from Equation (11). And as in the previous two cases, we end up with a non-linear equation to solve, from which the response at any DOF (a, b or c) can be computed. Besides, we note that the formulation is not different from the cases considered earlier with the only difference being that there are more filter coefficients to be computed in the present case.

4. COMPUTER SIMULATIONS

In this section computer simulations regarding a SDOF mechanical system with the applied force and non-linearity at the same DOF are shown. A SDOF system, shown in Figure 6, is chosen because of its simplicity regarding the associated computer simulations. The ramp invariance method computation is based on the digital filter formulation exposed in Section 3 (a), while the step invariance method computation is based on the recursive algorithm of the state transition matrix method as given in [3]. The objective is to compare the performance of the ramp invariance method against the step invariance method in relation to their dependence on the sampling frequency. As mentioned in Section 3, both methods compare favorably with each other in terms of computation times, but it should be pointed out that one is always looking for digital algorithms, which firstly behave well with change in the sampling frequency and secondly algorithms for which a low sampling frequency can be used, which has the beneficial effect of reducing computation times.

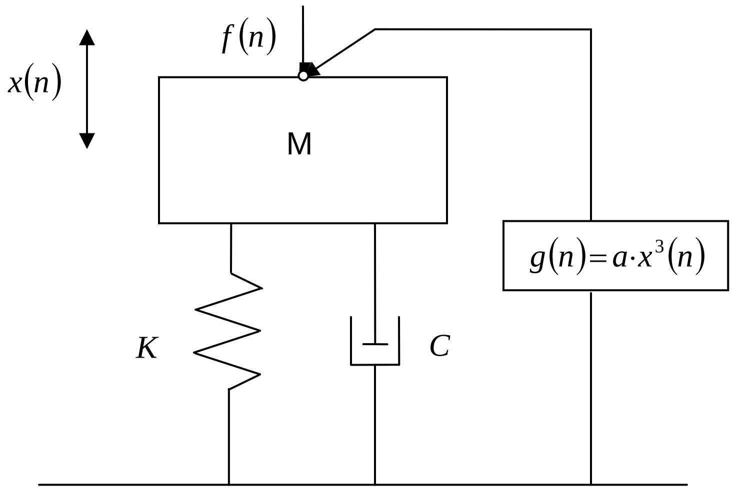

Figure 6. SDOF mechanical system with applied force and a cubic nonlinearity function (with a as constant) at the same DOF.

For comparison purposes, it has been found practical to introduce an error measure to quantify the behavior of the time displacement response with change in sampling frequency. The SDOF system shown in Figure 6 cannot be solved by any known analytical methods to the authors' best knowledge and a viable error measure can work on the premise that the response of the system gets more accurate with increasing sampling frequency. A relative error measure has thus been adopted and can also be viewed as an indicator of the degree of robustness of a digital method with the sampling frequency. This relative error measure is given by

$$\text{Relative error} = \frac{\underset{all\,n}{std}\left\{x(n)_{f=f_s} - x(n)_{f=\max f_s}\right\}}{\underset{all\,n}{std}\left\{x(n)_{f=\max f_s}\right\}} \tag{16}$$

where std is the Standard Deviation operator and f_s denotes the sampling frequency.

For the SDOF system shown in Figure 6, the equation of motion for the system is given by

$$M\,\ddot{x}(n)+C\,\dot{x}(n)+K\,x(n)+a\cdot x^3(n)= f(n) \tag{17}$$

The parameters used in the simulation were $M=1$ kg, damping ratio $\zeta=1\%$, $a=1\times10^8$, a natural frequency of $f_0=10$ Hz and the input force $f(n)$ is random band limited in the range of 0 to 20 Hz. The simulation was carried out in MATLAB® [5] on a IBM ThinkPad® Laptop equipped with a Pentium IV processor with 512 MB of physical RAM using a WIN XP® platform and operating at 1.8 GHz. Figure 7 shows the relative error (in percent) for the ramp invariance and step invariance methods for 50 runs. Besides, it was found convenient for simulation purposes to use 25 seconds of data. As a guide regarding the computational speed associated with these digital filter methods, 1 run (for one method) with a data size of 100 ksamples took 5.3 seconds.

It is to be noted that the choice of the constant parameter a gives us displacement responses from both methods which are of comparable magnitude to the input force f in terms of their standard deviations, implying that a strong non-linearity has been considered in the simulations.

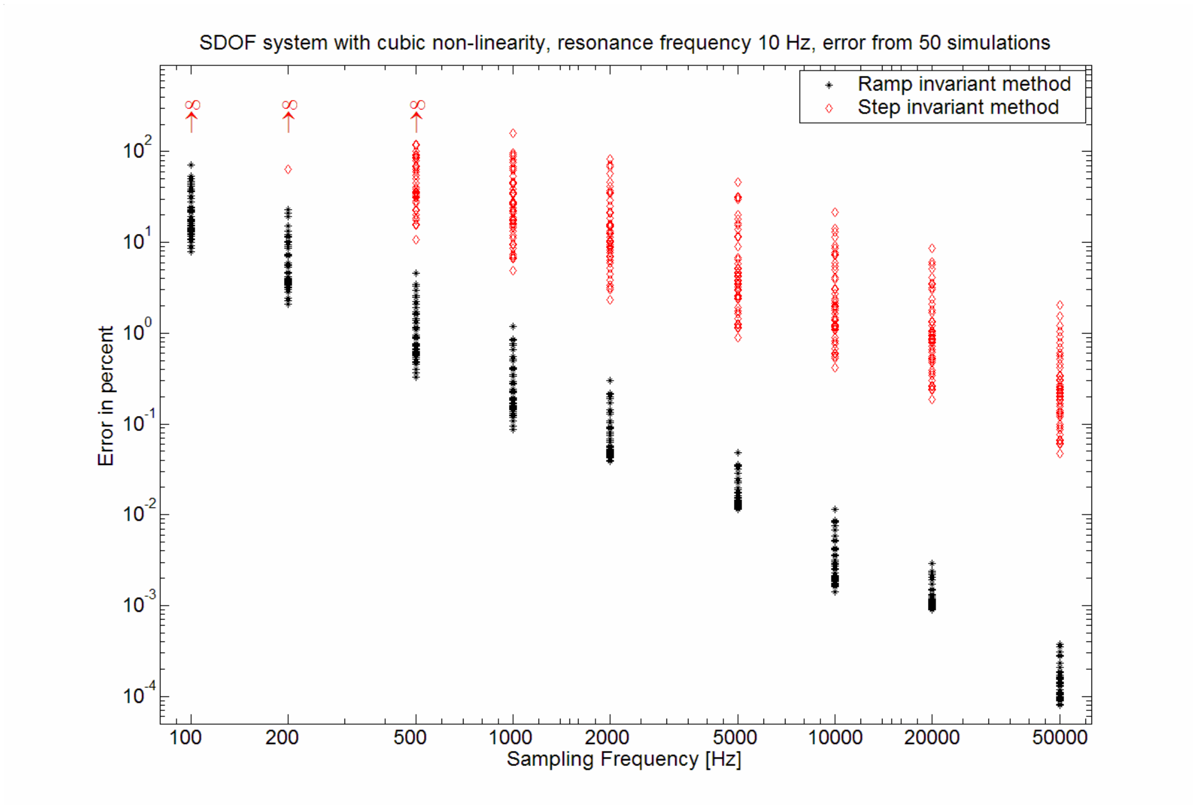

Figure 7. Relative error in percent for the ramp invariance and step invariance methods as a function of the sampling frequency from a 50-run simulation.

Three things can be readily observed from Figure 7: Firstly, the ramp invariance method behaves very well with the sampling frequency as regard to the relative error amplitude; secondly, there is less spread in the error for the 50 runs with the ramp invariance method at any given sampling frequency; and thirdly the algorithm for the step invariant method is unstable at low sampling frequencies.

CONCLUSIONS AND FURTHER WORKS

In this paper, a formulation of a digital filter method for computing the forced response of a MDOF mechanical system incorporating a single non-linear component has been proposed. Digital filter methods are superior in many ways as compared to common time domain integration techniques. Of the 3 common digital filter methods mentioned in this paper, the impulse invariance method because of its major drawback due to the aliasing phenomenon has not been considered in the computer simulations in Section 4. An error measure has also been proposed to quantitatively describe the performance of the ramp invariance and step invariance methods. These aforementioned digital filter methods are of practical interest, considering the high computing speeds associated with them and also they compare favorably well to each other in this aspect. The computer simulations demonstrated the supremacy of the ramp invariance method over the step invariance method in connection with its robustness to the sampling frequency.

The technique developed in this paper targeted only SDOF or MDOF systems with one non-linearity only and it can be applied to both zero-memory and finite-memory non-linear systems. Of Interest then is to extend it to the case where we have multiple non-linearities in a system at same or different DOFs.

In the recent years, lots of efforts have been put in the process of characterizing and modeling non-linear mechanical systems. In the task of structural modification, for instance, mathematical models serve their purpose only when they can be validated against actual measurements obtained from a structure. Some of the methods under study are ill-conditioned, thereby calling for large amounts of data to get reliable results, hence the need of fast and accurate simulation methods.

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